Visualizing Uncertainty with Fuzzy Rose Diagrams

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***Abstract*—This paper presents a novel method for visualizing vectors of fuzzy numbers. The proposed approach is an extension of the standard polar area diagram and can be applied to a single uncertain vector or a fuzzy weighted graph with vectors of fuzzy attributes on the vertices and/or edges. The resulting diagrams are intuitive to understand and do not require an extensive background in fuzzy set theory. By visualizing uncertain vectors in this way, the viewer can easily compare and contrast sets of fuzzy numbers. This can be useful in the context of decision support systems, particularly those involving multi-criteria decision making. We demonstrate our approach on the problem of finding a least-cost path through an uncertain environment.**

Keywords—visualization; decision support; fuzzy weighted graphs; multi-criteria decision making; fuzzy rose diagram

I. Introduction

There are many application domains in which information must be conveyed to a viewer quickly and concisely. Charts, graphs, maps, images, and other types of visualizations provide ways of organizing and presenting data in meaningful and informative ways. The common saying, "a picture is worth a thousand words" is indicative of the much higher information density that a graphic can provide over written text or numerical tables [1]. Furthermore, visualizations can summarize data, reveal hidden patterns, and make information more accessible to audiences with low numeracy [2]. Often, there is some uncertainty associated with the data that must also be represented visually. There is no consensus on the best way to convey uncertainty visually, although there are some established techniques that are appropriate for certain situations [3]. In the case of statistical variance, a plotted value can be shown with error bars or contour lines, depending on the context. In other situations, such as when dealing with vectors of fuzzy numbers, the process of visualizing the uncertainty may not be straightforward. The conventional method of plotting individual fuzzy membership functions, either on separate or combined axes, might not be the best way to convey the desired information. For example, radar charts [4] and disk diagrams [5] offer alternate ways of visualizing some types of fuzzy information. We develop the method presented in this paper as a way to visualize vectors of fuzzy numbers.

A fuzzy number is a convex, normalized fuzzy set $X: \mathbb{R} \rightarrow [0,1]$ that provides a way of representing uncertainty in the value of a real number [6]. They are particularly useful for capturing linguistic uncertainty, such as "about 2" or "a little more than 5". In this paper, we focus solely on non-negative fuzzy numbers that can be considered quantities of some feature or attribute. The membership function $\mu_X(x)$ gives the degree of membership that a specific value x has in the fuzzy number X. We define the support of the membership function as the interval for which $\mu_X > 0$, $[s_X^-, s_X^+]$. In order to visualize a fuzzy number, we establish the principle of perceptual proportionality, which states that the apparent size of a fuzzy quantity should be proportional to its value. Since fuzzy numbers cannot, in general, be represented by a single crisp value, we have developed a method for drawing shapes that convey the uncertainty in a fuzzy number, while allowing the actual value to be represented by its apparent size.

In this paper, we present a novel method for visualizing vectors of fuzzy numbers using an extension of the rose diagram [7], which can be further extended to display fuzzy weighted graphs [8]. The proposed approach allows such vectors to be easily compared and used to guide human-led decision making processes. An example of this is the determination of a least-cost path through a partially known environment. In [9], we approach this problem using the principle of bounded rationality, using an alpha-level OWA operator to imitate how a human decision-maker might only be able to consider a few factors at a time. The ability to visualize large quantities of uncertain information and to reason with the resulting information could be of great use in practice.

The remainder of this paper is organized as follows. Section II reviews the standard polar area diagram. Section III introduces our extension, the fuzzy rose diagram. Section IV shows how our method can be extended to show fuzzy weighted graphs. Section V gives an example of how our approach can be used in the decision-making problem of finding a least-cost path through an uncertain environment. Finally, our conclusions and ideas for future work are given in Section VI.

II. Polar Area Diagrams

A standard rose diagram is a polar area diagram, first made popular by Florence Nightingale as a way to show how the death toll in an unsanitary Turkish hospital changed over the course of two years [10]. They are sometimes referred to as coxcombs and provide a way to show a histogram on a periodic domain. Given a vector of crisp feature values $\boldsymbol{X} = \langle x_1, \ldots, x_n \rangle$, each feature is shown as a wedge arranged around the center of a polar plot. Each wedge has the same central angle, but the radius of each wedge differs to indicate the corresponding feature value. This is in contrast to a common

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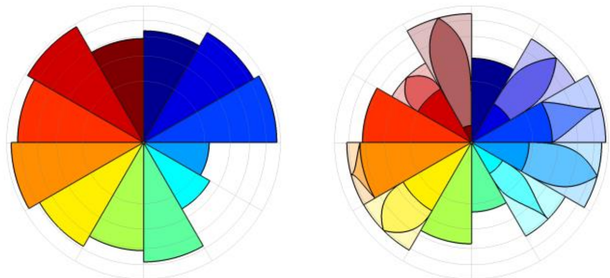


(a) (b)

Fig. 1. (a) Example of a standard rose diagram. (b) A fuzzy rose diagram.

pie chart, which uses a constant radius and adjusts the central angle of each wedge to indicate the relative values. Although rose diagrams are typically used to show how the quantity of some value changes with direction or time, they can also be used as a compact way to display multiple unrelated features or categories.

The guiding principle of the rose diagram is to adjust the radius of each wedge so that its area is proportional to its value. A direct mapping of the feature values to radius lengths would distort the final image, giving more area to larger values farther from the center and making small values appear disproportionately tiny. Since the objective is to provide an image that is perceptually proportional with respect to the features, the radius of each wedge is computed as

$$r_i = \sqrt{\frac{N x_i}{\pi}}, \tag{1}$$

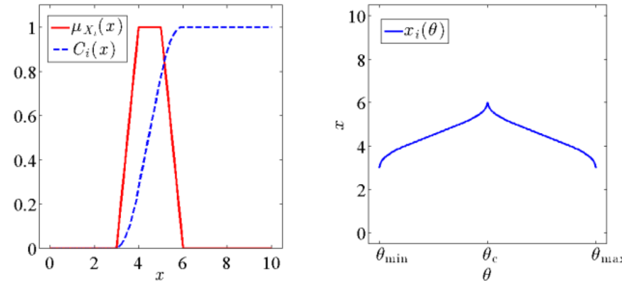
where x_i is the feature value and N is the number of features or wedges. In the case when the input features are crisp, the radius remains constant for the entire wedge. However, if the features are fuzzy numbers, the radius length must reflect the additional uncertainty. There are several approaches to this problem, which we will discuss in the next section. An example of a crisp rose diagram and our proposed fuzzy rose diagram is given in Fig. 1.

III. Fuzzy Rose Diagrams

We develop the fuzzy rose diagram as an extension of the crisp rose diagram that can display a vector of uncertain features represented as fuzzy numbers $\tilde{\boldsymbol{X}} = \langle \tilde{X}_1, \dots, \tilde{X}_n \rangle$ in a clear and compact manner. We aim to uphold the principle of perceptual proportionality so that large values will appear to have more area than small values, even with the added uncertainty. To accomplish this, we first describe two alternate approaches that build up to our final method. Examples of the three methods on various sets of fuzzy numbers are given in Fig. 3. Each row shows a different way of representing one of the fuzzy number sets.

A. Alpha-mapped Arcs

In this first naïve approach, we draw only the arc of each feature and do not fill in the area within the wedges. The arcs are each drawn with thicknesses equal to the widths of each feature's support. Rather than a solid black or colored line, the



(a) (b)

Fig. 2. (a) An example fuzzy number with its normalized cumulative membership function. (b) The resulting radius length over the span of the wedge for the cumulative petal method.

alpha value changes with distance from the origin to indicate the feature's membership function value. For feature i, the alpha value of the arc at radius r is defined as

$$F_i(r) = \mu_{X_i}(r). \tag{2}$$

An alpha value of 1 indicates a completely opaque arc and 0 indicates full transparency, allowing the background color to come through. Values between 0 and 1 are shown with varying degrees of transparency to indicate the strength of the membership function. Note that in this method the radius corresponds directly to the plotted value with no scaling. This is because arc lengths, rather than areas, are used to show relative magnitudes and the arc length of a wedge is directly proportional to the radius. A reference grid can be included to indicate the scale of the plot.

While this approach may be intuitive and easy to create, it abandons the notion of proportional areas. In Fig. 3 row (a) it can be seen that the most prominent features are those with the most uncertainty. Crisp values are drawn as a single line, which can be difficult to see and may even be missed altogether. Large values have empty areas near the plot center which seems counterintuitive and distorts the perception of information. We include it here for its utility in developing the following methods and as a comparison to highlight their advantages.

B. Cumulative Fuzzy Wedges

In this approach, we address one of the major drawbacks of the previous method and aim to create filled wedges that uphold the concept of proportional areas. We still use alpha-mapped arcs as before, but with alpha values that only decrease with additional distance from the origin. In this way, we can show completely opaque areas for values less than the minimum of the support and full transparency for values greater than the support maximum. The values within the support are shown with alpha values between 0 and 1, transitioning from opaque to transparent as the radial distance increases.

The rationale for this approach is to think of each fuzzy feature as some unknown quantity, say the amount of water in a glass, for example. The membership function defines the degree to which each possible crisp value represents this feature. If the amount of water in a glass is defined by a

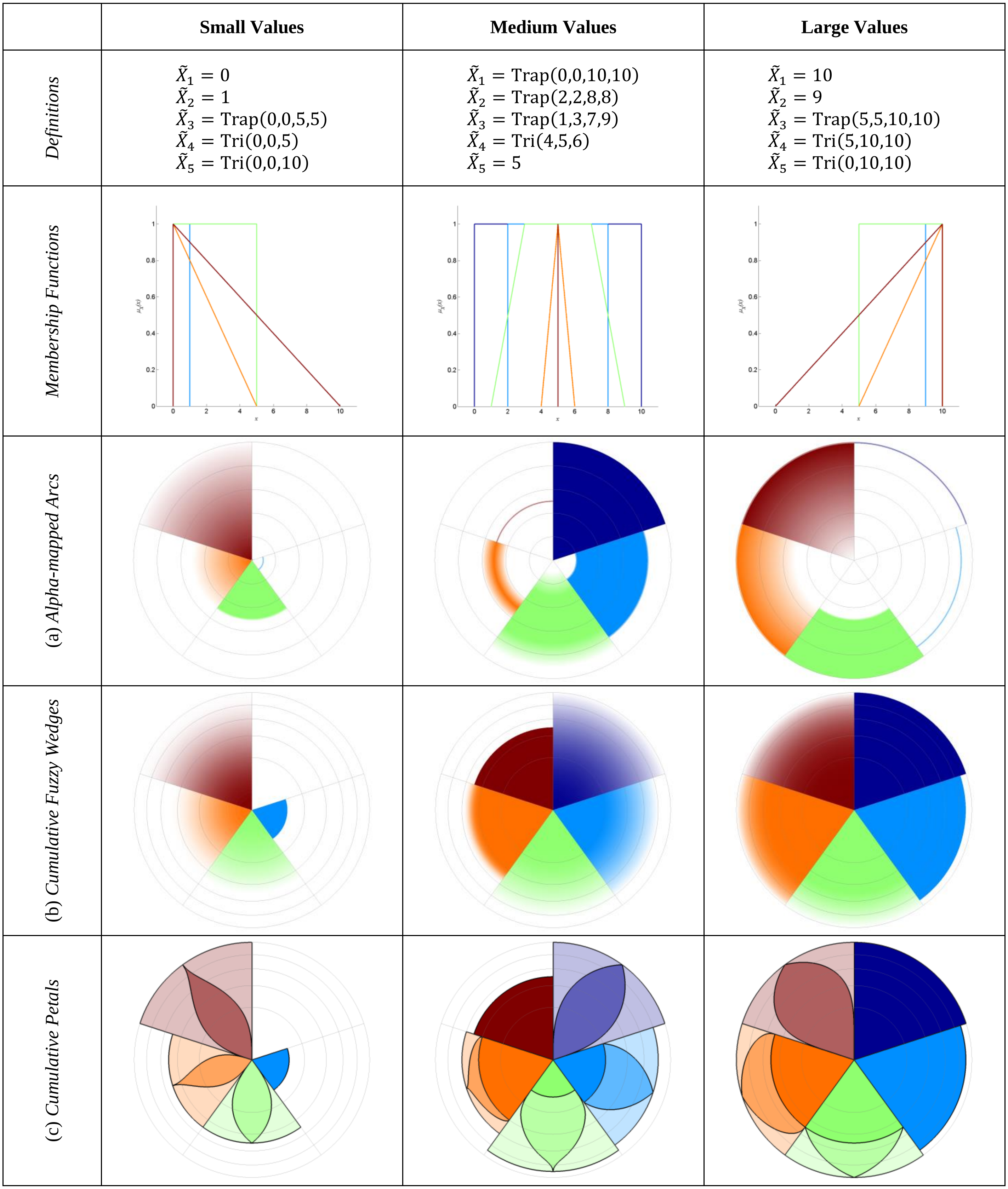


Fig. 3. Examples of fuzzy rose diagrams drawn in three different styles. Each column represents a different set of fuzzy numbers represented in various ways. Each method has its own set of strengths and weaknesses. The membership functions plotted directly clearly show the function values, but are all overlapping and do not have proportional areas. The alpha-mapped arcs method (a) does not have proportional areas, and makes crisp values difficult to see. The cumulative fuzzy wedges method (b) and cumulative petals method (c) both preserve proportional area, with cumulative petals achieving this effect without relying on alpha-level mapping.

triangular membership function with a peak of 100 mL and a support of [50 mL, 150 mL], we would expect there to be at least 50 mL of water in the glass and no more than 150 mL. For any given value x within this range, we wish to know the confidence that the glass has at least x amount of water. This is easily computed using the normalized cumulative membership function, defined for a fuzzy number $\tilde{X}_i$ as

$$C_i(x) = \frac{1}{A_i} \int_0^x \mu_{X_i}(u)\, du, \tag{3}$$

where A_i is the total area under the membership function, defined as

$$A_i = \int_0^\infty \mu_{X_i}(u)\, du. \tag{4}$$

The normalized cumulative membership function is a non-decreasing function with $C_i(x) = 0$ for $x \le s_{X_i}^-$, and $C_i(x) = 1$ for $x \ge s_{X_i}^+$, where $\left[s_{X_i}^-, s_{X_i}^+\right]$ is the support of $\tilde{X}_i$. This is shown in Fig. 2 (a). We can get the corresponding alpha level value for a particular x value by subtracting this function from one. In order to maintain proportional areas and reduce distortion, we solve Equation 1 for x_i giving the value that should be used at a particular radius length. The final alpha value is given as

$$F_i(r) = 1 - C_i(x) \tag{5}$$

where

$$x = \frac{\pi r^2}{N}. \tag{6}$$

This approach creates a pleasing image that is perhaps the easiest to interpret simply by looking at the resulting image (Fig. 3 row (b)). Area proportionality is preserved, making large values clearly visible and reducing the apparent size of small values. The uncertainty is represented by the "fuzziness" of the arc edge. Crisp values have a well-defined boundary, whereas more uncertain values have a wider gradient. While this may be easy to understand with little to no training, the fuzzy boundaries make it difficult to discern precise values. Also, the figure quality is highly dependent on the ability of the medium to accurately depict gradients and the interpretation is open to human subjectivity. We offer an alternative method in the next section to ameliorate these shortcomings.

C. *Cumulative Petals*

While the previous method captures the notion of fuzzy feature values and maintains proportional areas, it requires the ability to both draw and interpret alpha-mapped images. There are several reasons why this may not be possible or desirable. Some mediums, such as black and white photocopies, may not be able to display gray scale images. Furthermore, the reproduction accuracy of the gradated image depends greatly on the quality of the display, projector, or print medium. One solution is to use dithering to reduce the number of required colors, representing different shades with different quantities or sizes of dots, such as with newspaper or magazine printing. An alternate approach is to use the final method we describe next.

This last method conveys the uncertainty associated with each feature without relying on alpha-mapped values. To do this, we plot the inverse normalized cumulative membership function directly for each feature. For each fuzzy number $\tilde{X}_i$, $C_i(x)$ is a function that maps values from the support $\left[s_{X_i}^-, s_{X_i}^+\right]$ into the range [0,1]. Values outside the support are set to either 0 if $x < s_{X_i}^-$ or 1 if $x > s_{X_i}^+$. The inverse of this function, $C_i^{-1}(y)$, maps values from the range [0,1] into the support. For each feature, we define the range of the central plot angle as $[\theta_{\min}, \theta_{\max}]$ and the mean of these two as θ_c. We divide the wedge into two symmetric halves and compute the values of the inverse normalized cumulative membership function as θ changes from $\theta_{\min}$ to $\theta_{\max}$,

$$x_i(\theta) = C_i^{-1}\left(\frac{2|\theta - \theta_c|}{\theta_{\max} - \theta_{\min}}\right). \tag{7}$$

This is shown in Fig. 2 (b). To compensate for the polar area distortion, we apply Equation 1 to get the actual plotted radius length,

$$r_i(\theta) = \sqrt{\frac{N x_i(\theta)}{\pi}}. \tag{8}$$

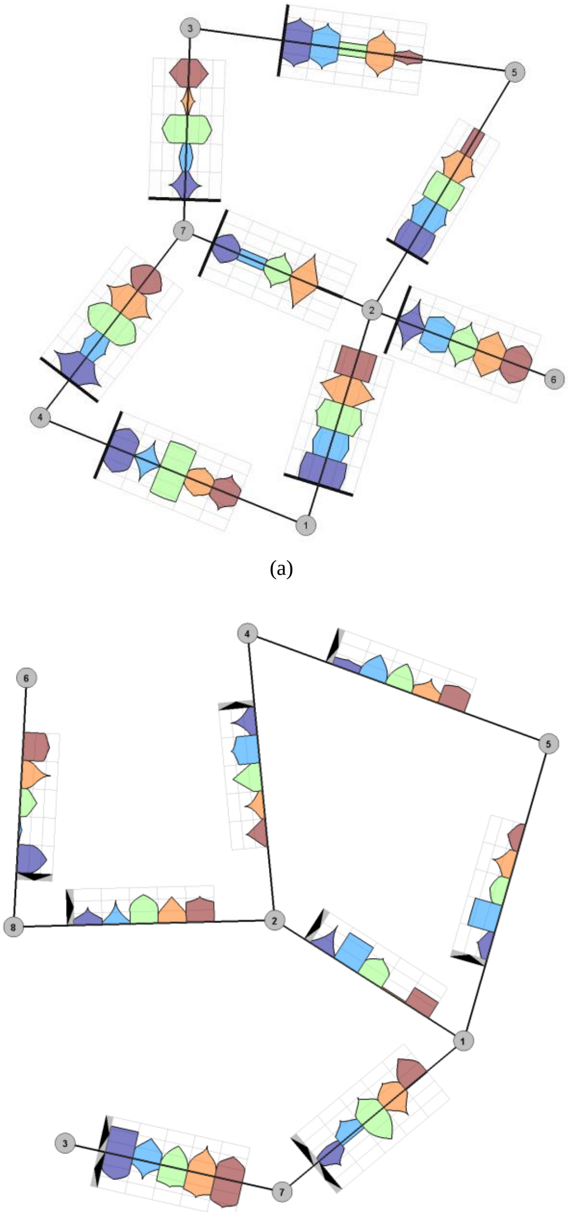
This produces a petal-shaped wedge that spans the entire width of the wedge at the support minimum, and peaks in the middle at the support maximum. To improve interpretability, we add an extra arc for both the support minimum and maximum values.

The resulting figure is a clear image with crisp lines, still capable of representing uncertain values (Fig. 3 row (c)). Multiple colors and transparency values can be used to enhance the image, but are not required for interpretation. In this method, the shape of the petal and the distance between the minimum and maximum arcs represents the uncertainty. Crisp values have no apparent petal shape and appear the same as in the previous method. Uncertain values are shown with various petal shapes, indicating the type of uncertainty. For example, the fuzzy number $\tilde{X}$=Tri(0,0,10) has a very narrow petal, which creates a small area and indicates that small values are more likely to be observed than large values. In contrast, the fuzzy number $\tilde{X}$=Tri(0,10,10) has the same support, but a much wider petal, which creates a larger area and suggests that large values are more likely. The original membership function could, in theory, be reconstructed from the petal shape by taking the derivative of the cumulative membership function. In practical terms, the width of the petal at a given radius is proportional to the confidence of observing at least that value.

IV. Fuzzy Weighted Graphs

The fuzzy rose diagram is useful for representing a single vector of fuzzy numbers, which is often sufficient for many applications. In this section, we show how the fuzzy rose diagram can be modified to display the weights on a fuzzy weighted graph. For vertex-weighted graphs, each vertex can simply be displayed as a separate fuzzy rose diagram. However, the more common edge-weighted graph requires some additional manipulation in order to display the fuzzy vector along a linear edge.

There are several approaches for representing the weights of an edge-weighted graph visually. One approach commonly



(a)

(b)

Fig. 4. Examples of randomly generated undirected (a) and directed (b) fuzzy weighted graphs.

used when there is only a single crisp weight on each edge is to modify the length of each edge to be proportional to its value. This affects the layout of the graph and may not be an option when the vertices have fixed locations, as in a geospatial environment. Another commonly used approach is to adjust the width or opacity of each edge. While this works well when there is only one weight per edge, it becomes difficult to apply with multiple weights.

We can create a linear version of the fuzzy rose diagram by plotting each feature as an equal-width segment along a linear axis. This is analogous to how a standard histogram plot could be created by "unwrapping" the rose diagram. Each segment is

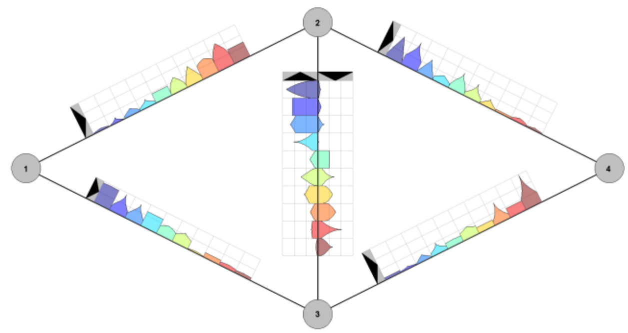


Fig. 5. Example of a fuzzy weighted directed graph with 10 uncertain features on each edge. There are four possible paths from vertex 1 to vertex 4: 1–2–4, 1–2–3–4, 1–3–2–4, and 1–3–4.

plotted as the inverse normalized cumulative membership function, with θ ranging from $\theta_{\min}$ to $\theta_{\max}$ over the span of the segment using Equation 7, as is shown in Fig. 2 (b). Because the values are drawn in a Cartesian coordinate system, there is no need to rescale the values before plotting, unless to achieve a uniform scale effect as will be described below. In order to improve the readability in small diagrams, the minimum and maximum lines may be omitted.

For both directed and undirected fuzzy weighted graphs, we draw a linear version of the fuzzy rose diagram along each edge with the same width W for each diagram. This ensures consistency in the apparent size of each fuzzy vector. If vertices are also weighted and drawn as fuzzy rose diagrams, then the scale of the edges should be consistent with that of the vertices. This can be accomplished by multiplying the values of the linear edge diagrams by an extra scale factor

$$\gamma = \frac{\lambda^2 N}{W}, \tag{9}$$

where λ is the overall scale factor applied to the fuzzy rose diagrams on the vertices. Determining an appropriate overall scale is a matter of some subjectivity, as too large of a scale will cause overlapping, whereas a scale that is too small may become difficult to read. The edge routing can be adjusted as well to facilitate the placement of the diagrams, although a detailed discussion on edge placement is beyond the scope of this paper. Our method works best on planar graphs with well-spaced vertices, allowing edges to be drawn as straight lines between vertices.

For undirected graphs, we choose to mirror the fuzzy vector diagrams on both sides of the edge lines as shown in Fig. 4 (a). Additionally, we include a grid and a vertical reference axis along one side of each diagram to show how the features are ordered. Because the diagrams can be drawn with any orientation, it is important to include a way to determine which feature is first. Color helps to distinguish features, but color alone should not be relied upon, in case accurate color reproduction is unavailable. We use the convention of placing the axis on the leftmost side of each diagram, although alternate orientations could be used if appropriate.

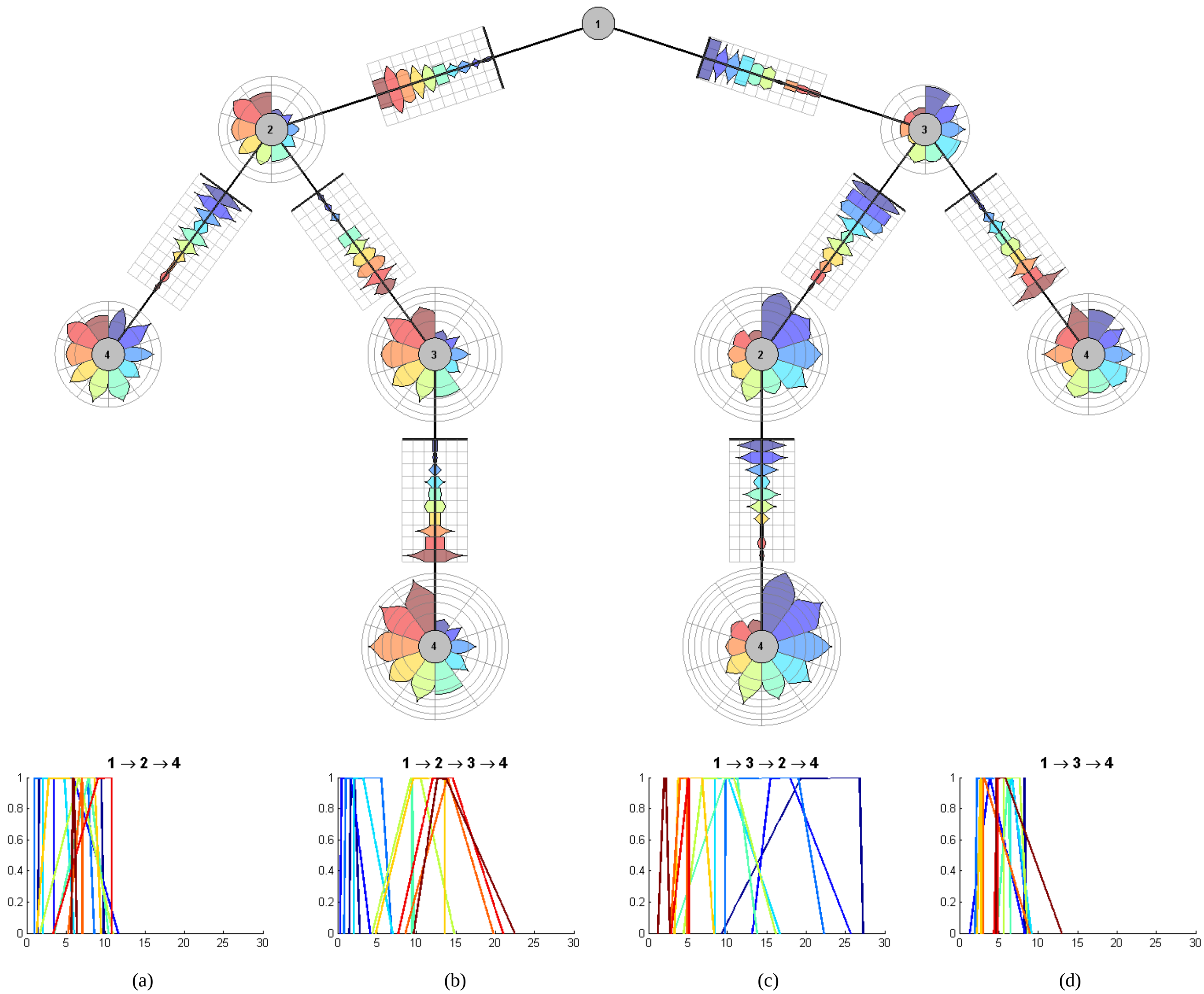


Fig. 6. Costs of the four paths through the example graph in Fig. 5 presented as a fuzzy rose decision tree. The vectors of fuzzy numbers along each path are summed together to create the fuzzy rose diagrams at each node. The fuzzy rose diagram at each leaf node represents the total sum of the vectors along that path. Below each leaf node, the final fuzzy rose diagrams are redrawn in the standard way by plotting the membership functions on a common set of axes.

For directed graphs, we plot the fuzzy vector diagrams on only one side of each edge. We use a clockwise notation, placing right-directed edges on top, left-directed edges on bottom, up-directed edges on the left, and down-directed edges on the right. For bidirectional edges, the features are aligned to be across from their opposite-direction counterparts. This allows the features in both directions to be compared more easily, and ensures that the reference axis is always on the same side for both directions of an edge. To indicate the direction of each diagram, the reference axis is drawn as either an inward or outward facing arrow according to the clockwise notation.

V. Fuzzy Rose Diagrams for Decision Support

The fuzzy rose diagram excels at displaying vectors of uncertain information in a compact and organized manner. This is particularly useful for decision-support systems involving human decision-makers. It may be difficult to understand the tradeoff between alternate choices without first having a clear picture of a high-dimensional and uncertain environment. Fuzzy rose diagrams allow several options to be presented and compared visually, giving a more complete understanding of the decision to be made.

As an example, consider the fuzzy weighted directed graph in Fig. 5. Suppose this graph represents some environment where vertices represent locations and edges represent the paths between them. Each path has a set of attributes that may only be known with some degree of certainty. For example, in [9] each path has a fuzzy number representing the perceived distance, elevation change, path quality, amount of shade, and difficulty of water crossings. We could also consider the amount of traffic, distance from some attractor or detractor, expected travel time, or any number of additional features. Consider the problem of a decision-maker planning a least-cost path from vertex 1 to vertex 4 in this graph. There are four possible paths: 1–2–4, 1–2–3–4, 1–3–2–4, and 1–3–4. Because there are many uncertain features on each edge and the decision-maker may weigh the features differently, it is not obvious which of these paths is best. To compare these options, we construct the decision tree shown in Fig. 6.

Fig. 6 shows the path-planning problem as a fuzzy rose decision tree. The root node corresponds to the starting point (vertex 1) in the graph. The four possible routes are shown as the branches that end with leaf nodes corresponding to vertex 4. We plot the tree as an undirected graph, since direction is implicit, with the reference axis on the side of the parent node. Each node is shown with a fuzzy rose diagram indicating the fuzzy sum of the features up to that node. In this example, we are using the fuzzy sum operator, based on the Extension Principle, which computes the sum of two fuzzy numbers using interval arithmetic for each alpha-cut. The nodes are shown with a solid center circle with a number to indicate the vertex at that point in the route. This effect is achieved by adding a constant value to the plotted radius lengths of each feature to compensate for the area hidden under the center circle. The reference grids must also be adjusted accordingly.

The fuzzy rose diagrams at the leaf nodes in Fig. 6 clearly show the differences between the route choices. Route (b) minimizes the first few features in blue, but maximizes the last features in red. Alternately, route (c) minimizes the last features, but maximizes the first features. Routes (a) and (d) offer more of a balance across all features, although they are clearly different. The fuzzy rose diagrams help a human decision-maker choose between a set of options. For example, if our only criterion were to minimize the 7th feature in yellow, we would choose route (d). Alternately, if we wanted to minimize the 1st feature without getting too much of any one feature, we might choose route (a) because although there is more uncertainty in this feature for route (a) than route (d), there is the possibility of it being smaller than the very certain value observed in route (d). The nearly constant radius length of some features indicates that they are known with a high degree of certainty, whereas other features with more uncertainty are shown with various petal shapes.

Below the leaf nodes in Fig. 6, we plot the resulting membership functions obtained by summing the vectors of fuzzy numbers along each route. These plots use the same information as the fuzzy rose diagrams drawn at each leaf node, but are arguably more difficult to interpret and to compare than their fuzzy rose counterparts. We can determine that routes (b) and (c) have some features that are larger than those of other routes, but it is difficult to see that they also minimize some features. The plots appear as a jumbled mess of lines with individual features distinguished only by color. Different line weights or patterns could be used to distinguish the different features, but this would not reduce the overall clutter. Plotting the features independently for each vector would improve readability, and would be better for a comparison where exact values are important, but would take up considerably more space and may not be as simple to interpret for an audience with low numeracy.

VI. Conclusion

We have shown how the fuzzy rose diagram can present vectors of fuzzy numbers in a compact yet descriptive manner. This can help a decision-maker choose between several options while evaluating the potential tradeoffs. Our method is also able to display fuzzy weighted graphs with multiple attributes on the graph edges and vertices. This can be used to depict uncertain environments and to display decision trees involving vectors of uncertain quantities. We have worked to design the fuzzy rose diagram to follow the principle of perceptual proportionality, which should allow those without a formal training in fuzzy set theory to understand the information. By using the shapes of the petals to represent uncertainty, we do not require extra colors or textures to represent the features, allowing color to simply enhance the interpretability of the image.

While not applicable in all circumstances, the fuzzy rose diagram has many potential uses and should help in the presentation of high-dimensional uncertain data. Ideas for future work include showing how a shortest-path algorithm can operate on a fuzzy weighted graph, creating interactive displays of real-time uncertainty, and investigating perceptual similarity measures between fuzzy rose diagrams.

References

[1] E. Tufte, *The Visual Display of Quantitative Information*. Cheshire, CT: Graphics Press, 2001.

[2] E. Peters, J. Hibbard, P. Slovic, and N. Dieckmann, “Numeracy skill and the communication, comprehension, and use of risk-benefit information.,” *Health Aff.*, vol. 26, no. 3, pp. 741–8, 2007.

[3] D. Spiegelhalter, M. Pearson, and I. Short, “Visualizing uncertainty about the future.,” *Science*, vol. 333, no. 6048, pp. 1393–400, Sep. 2011.

[4] V. Atanassova, “Representation of Fuzzy and Intuitionistic Fuzzy Data by Radar Charts,” *Notes on Intuitionistic Fuzzy Sets*, 2010.

[5] Y. Park and J. Park, “Interactive Visualization of Fuzzy Set Operations,” in *IS&T/SPIE Electronic Imaging*, 2010.

[6] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*. Upper Saddle River, NJ: Prentice-Hall, 1995.

[7] P. J. Lewi, “Florence Nightingale and Polar Area Diagrams,” in *Speaking of Graphics*, [ebook] http://www.datascope.be/sog.htm, 2006.

[8] C. Cornelis, P. De Kesel, and E. E. Kerre, “Shortest paths in fuzzy weighted graphs,” *Int. J. Intell. Syst.*, vol. 19, no. 11, pp. 1051–1068, Nov. 2004.

[9] A. R. Buck, J. M. Keller, and M. Popescu, “An alpha-level OWA implementation of Bounded Rationality for Fuzzy Route Selection,” in *World Conference on Soft Computing*, 2013.

[10] F. Nightingale, Notes on Matters Affecting the Health, Efficiency and Hospital Administration of the British Army, Founded Chiefly on the Experience of the Late War. London: Harrison, 1858.