On the Ternary Spatial Relation "Between"

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Our Goal

- We are interested in answering two questions:
  - *What is the region of space located between two objects A_1 and A_2?*
  - *To what degree is B between A_1 and A_2?*

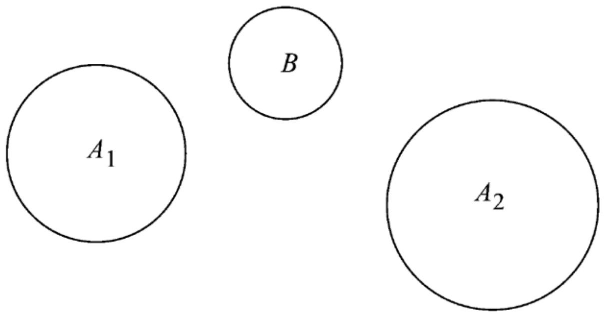


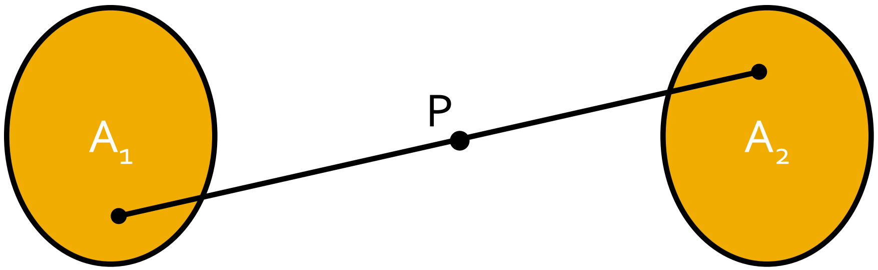
Fig. 1. Is the object B between A_1 and A_2, and to which degree?

Outline

- Existing Definitions
- Convex Hull
- Morphological Dilations
- Visibility
- Satisfaction Measure
- Properties
- Examples

Existing Definitions

- Merriam-Webster Dictionary defines between as "*In the time, space, or interval that separates.*"
- Some crisp approaches:
  - A point is between two objects if it belongs to a segment with endpoints in each of the objects.



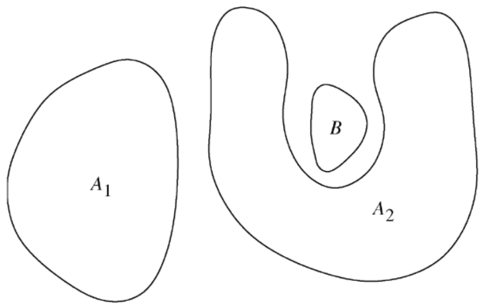


Fig. 2. Example where B is considered to be between A_1 and A_2 according to the definition of [9].

Existing Definitions

- Another crisp approach:
  - Using bounding spheres, B is between A_1 and A_2 if it intersects the line between the centroids of A_1 and A_2.

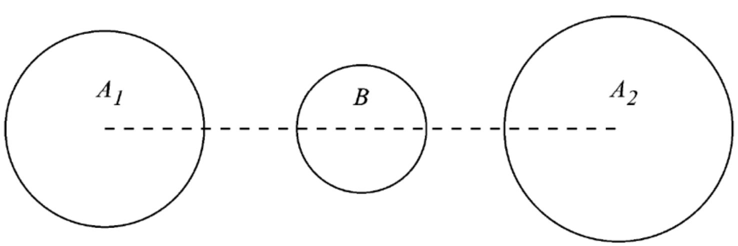


Fig. 3. Sphere B is between spheres A_1 and A_2 [11].

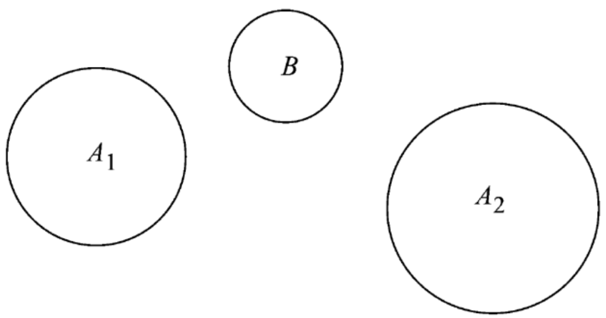


Fig. 1. Is the object B between A_1 and A_2, and to which degree?

Existing Definitions

- Fuzzy approaches:
  - For all $a_1 \in A_1, a_2 \in A_2, b \in B$ calculate the angle θ at b between the segments $[b, a_1]$ and $[b, a_2]$. Define a function $\mu_{\text{between}}(\theta)$ to measure the degree to which b is between a_1 and a_2.

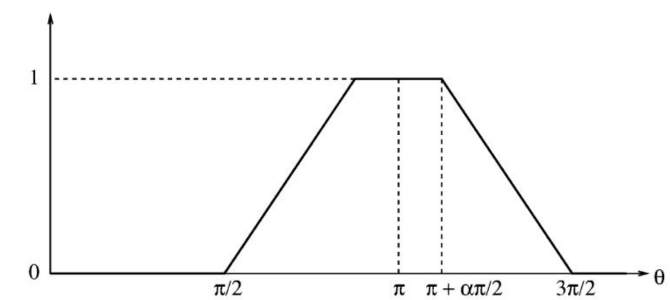


Fig. 4. Illustration of the function $\mu_{\text{between}}(\theta)$ proposed in [12] (α is a parameter expressing the tolerance in the idea of between).

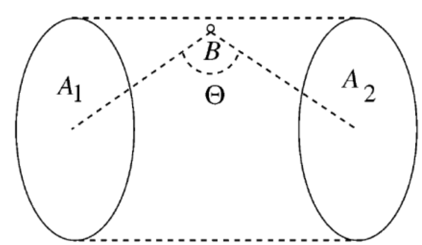


Fig. 5. Example where the definition of [12] hardly corresponds to intuition (Θ corresponds to the average angle and is significantly smaller that π, while B would be intuitively considered completely between A_1 and A_2).

[12] R. Krishnapuram, J. M. Keller, and Y. Ma, "Quantitative Analysis of Properties and Spatial Relations of Fuzzy Image Regions," *IEEE-Trans. Fuzzy Syst.*, vol. 1, no.3, pp. 222-233, Feb. 1993.

Existing Definitions

- Using the histograms of forces:
  - The spatial relationship between two objects can be modeled with force histograms, which give a degree of support for the statement, "A is in direction θ from B."

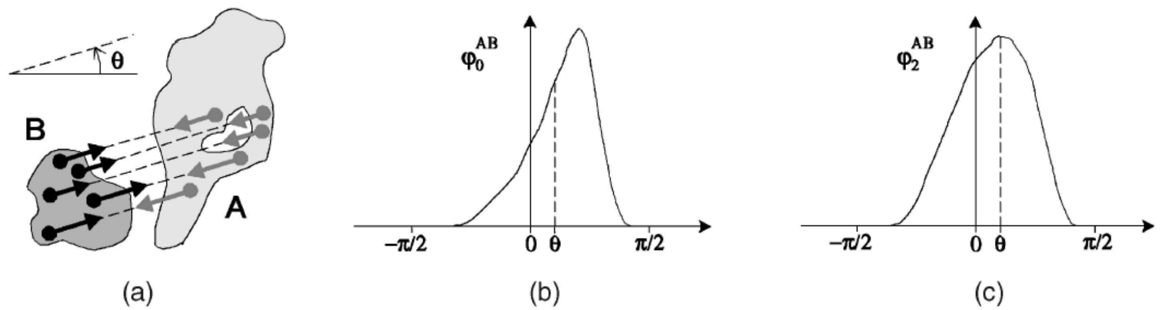


Fig. 6. Force histograms. Physical interpretation and examples. (a) $\varphi^{AB}(\theta)$ is the scalar resultant of elementary forces (black arrows). Each one tends to move B in direction θ. (b) The histogram of constant forces associated with (A, B) is one possible representation of the position of A relative to B. (c) The histogram of gravitational forces associated with (A, B) is another possible representation.

Existing Definitions

From P. Matsakis and S. Andréfouët, "The Fuzzy Line Between Among and Surround," in *Proc. IEEE FUZZ*, 2002, pp. 1596-1601.

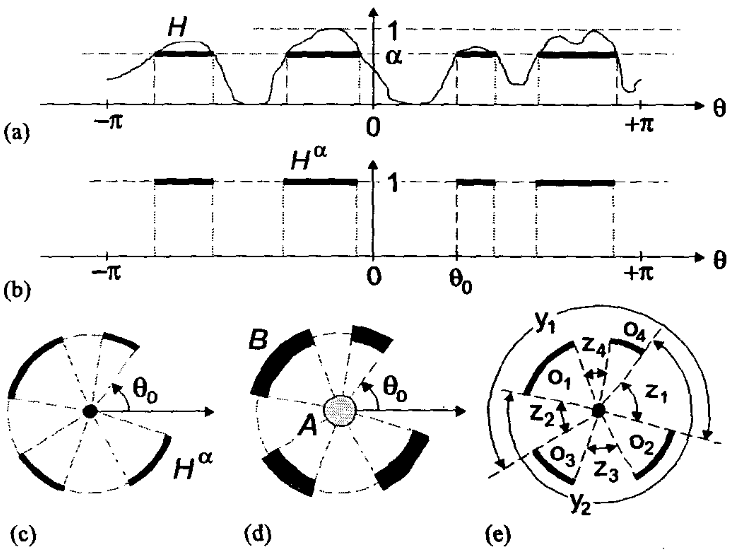


Fig. 4. (a) The normalized force histogram H is assimilated to a fuzzy set. (b) H^α is the cut of level α. (c) Polar representation of H^α. We would get a very similar diagram if the objects A and B were as in (d). (e) Angles associated with H^α.

$$q=1 \Rightarrow b^\alpha(A,B)=0$$

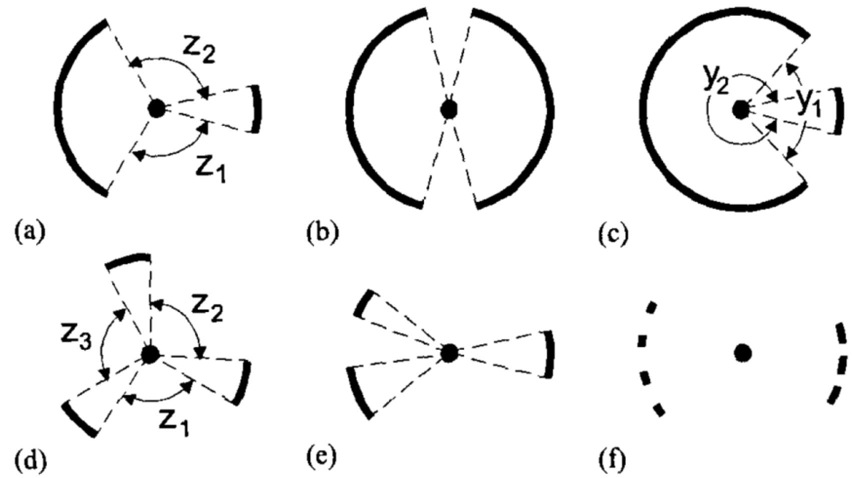


Fig. 6. Degree to which H^α describes a "between" situation.
(a) $b^\alpha(A,B)=1$. (b) $b^\alpha(A,B)=1$. (c) $b^\alpha(A,B)\approx 0.4$.
(d) $b^\alpha(A,B)=0$. (e) $b^\alpha(A,B)\approx 0.5$. (f) $b^\alpha(A,B)\approx 0.8$.

$$q=2 \Rightarrow b^\alpha(A,B)=min\left(1, k\frac{z_2}{z_1}, k'\frac{y_1}{\pi}, k'\frac{y_2}{\pi}\right)$$

$$q>2 \Rightarrow b^\alpha(A,B)=min\left(1, k\frac{z_2}{z_1}, k'\frac{y_1}{\pi}, k'\frac{y_2}{\pi}, max\left(0, 1-k''\frac{z_3}{z_2}\right)\right)$$

Existing Definitions

- Limitations of this approach:
  - Considers the union of objects, rather than their individual areas.

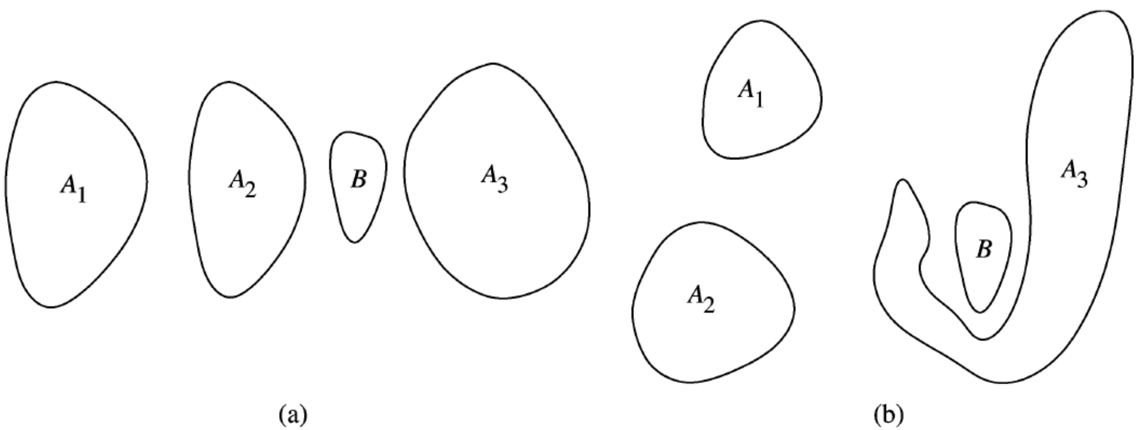


Fig. 6. Illustration of the definition of [13]. (a) A case of ambiguity: $A = A_1 \cup A_2 \cup A_3$ and B are considered to satisfy the relation according to this definition, while B is between A_2 and A_3, but not between A_1 and A_2. (b) A case with a nonvisible concavity where, again, the relation is satisfied according to this definition with a nonzero degree.

Convex Hull

- For any set X, its complement X^C, and its convex hull $CH(X)$, we define the region of space between objects A_1 and A_2 as $\beta(A_1,A_2)$.

$$\beta_{CH}(A_1, A_2) = CH(A_1 \cup A_2) \cap A_1^C \cap A_2^C = CH(A_1 \cup A_2) \setminus (A_1 \cup A_2)$$

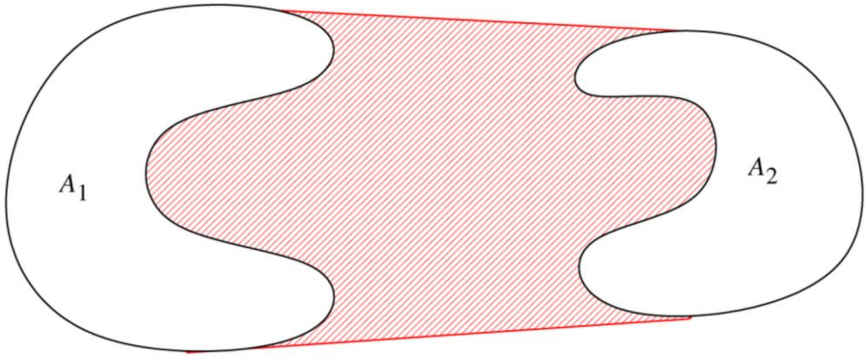


Fig. 9. Definition from convex hull: The dashed area corresponds to $\beta(A_1, A_2)$. (Color version available online at http://ieeexplore.ieee.org.)

Convex Hull

- Components of $\beta(A_1,A_2)$ which are not adjacent to both A_1 and A_2 should be suppressed.
- However, this leads to a continuity problem:

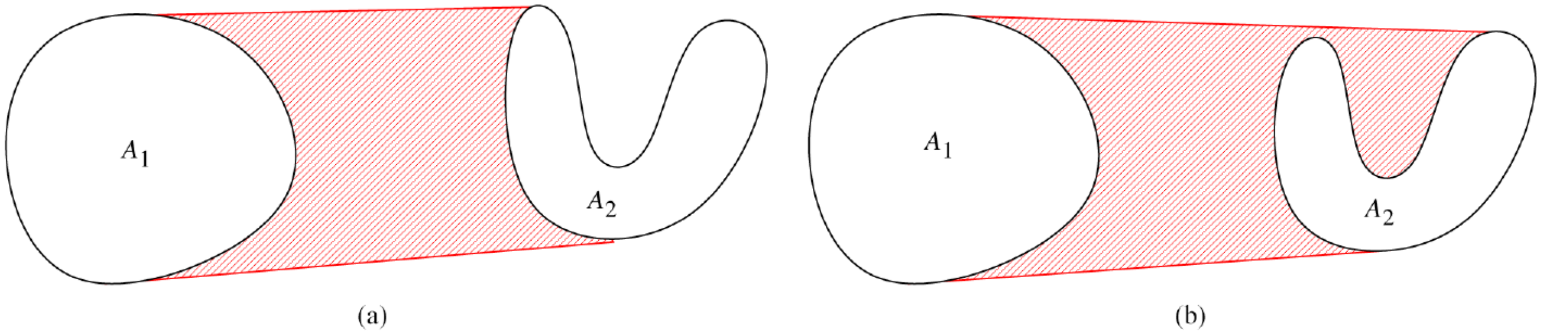


Fig. 10. Continuity problem: A_2 can be deformed continuously from situation (a) to situation (b), but the region between A_1 and A_2 does not vary continuously. (Color version available online at http://ieeexplore.ieee.org.)

Convex Hull

- Extension to the fuzzy case:

$$\beta_{CH}(A_1, A_2) = CH(\mu_{A_1} \cup \mu_{A_2}) \cap c(\mu_{A_1}) \cap c(\mu_{A_2})$$

  - Recommended to chose a t-norm that satisfies the law of excluded middle, such as Lukasiewicz' t-norm $\forall (a,b) \in [0,1],\ t(a,b) = \max(0, a+b-1).$

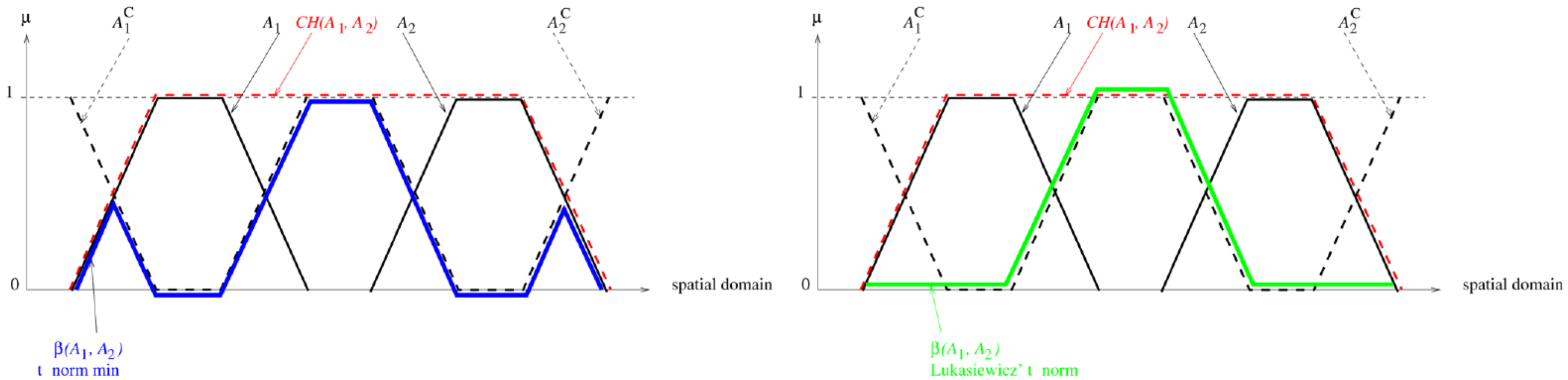


Fig. 11. Computing $\beta(A_1, A_2)$ from the convex hull of the union of fuzzy objects A_1 and A_2 with the minimum t-norm (thick blue line on the left) and with Lukasiewicz' t-norm (thick green line on the right). When using the min, triangles appear on the left and on the right, which are unwanted since they can hardly be considered being between A_1 and A_2. This phenomenon does not occur with Lukasiewicz' t-norm. (Color version available online at http://ieeexplore.ieee.org.)

Non-connected Objects

- If A_1 can be decomposed into connected components $\bigcup_i A_1^i$ we can define the region between A_1 and A_2 as $\beta(A_1, A_2) = \beta(\bigcup_i A_1^i, A_2) = \bigcup_i \beta(A_1^i, A_2)$ and similarly if both objects are non-connected.
  - This is better than using $CH(A_1,A_2)$ directly

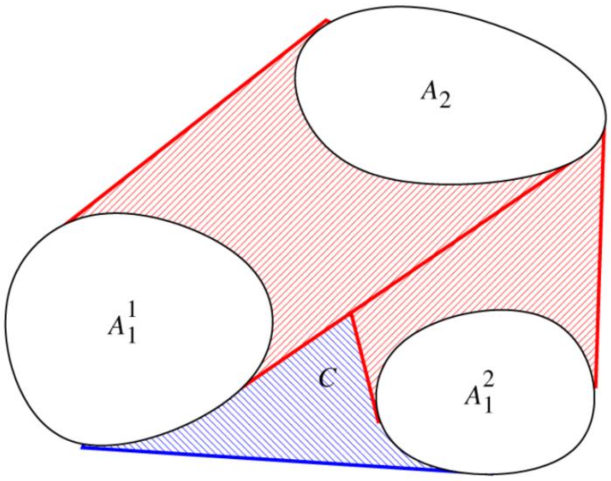


Fig. 12. Region C belongs to $CH(A_1 \cup A_2)$, but not to $CH\,(A_1^1 \cup A_2) \cup CH\,(A_1^2 \cup A_2)$, while an object in C would not be considered between $A_1 = A_1^1 \cup A_1^2$ and A_2. (Color version available online at http://ieeexplore.ieee.org.)

Morphological Dilations

- Definition based on Dilation and Separation:
  - Find a "seed" for $\beta(A_1,A_2)$ by dilating both objects until they meet and dilating the resulting intersection. $\beta_{Dil}(A_1,A_2) = D^n\left[D^n(A_1)\cap D^n(A_2)\right]\cap A_1^C \cap A_2^C$ where D^n denotes a dilation by a disk of radius n.

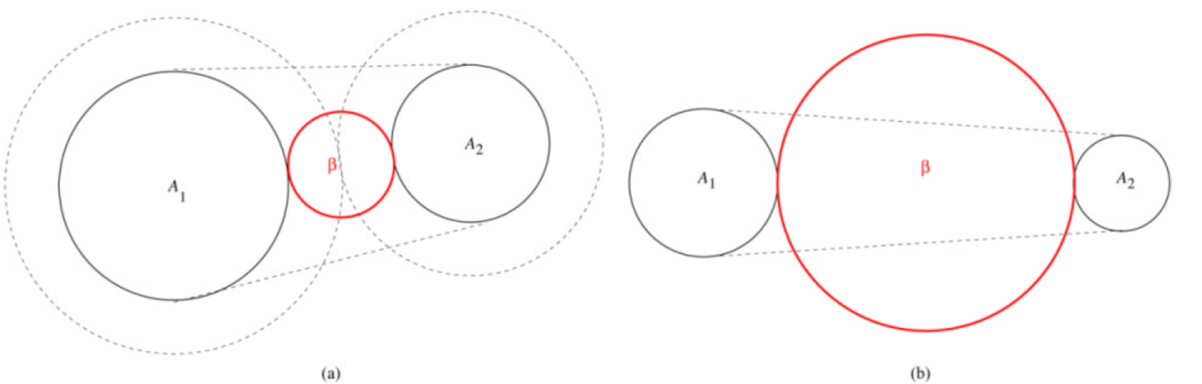


Fig. 13. Dilation of the intersection of the dilations of A_1 and A_2 by a size equal to their half minimum distance. The obtained region for β may exclude some parts belonging to the convex hull of (dashed straight lines) $A_1 \cup A_2$ (a) in some cases, or (b) may be too extended. (Color version available online at http://ieeexplore.ieee.org.)

Morphological Dilations

- Using watersheds and SKIZ (skeleton by influence zones):
  - Reconstruction of $\beta(A_1,A_2)$ by geodesic dilation.

$$\beta_{sep}(A_1,A_2) = D^{\infty}_{CH'(A_1 \cup A_2)}(WS) = D^{\infty}_{CH'(A_1 \cup A_2)}(SKIZ)$$

Where $CH'(A_1 \cup A_2) = CH(A_1 \cup A_2) \setminus (A_1 \cup A_2)$ from which connected components of the convex hull not adjacent to both sets are suppressed.

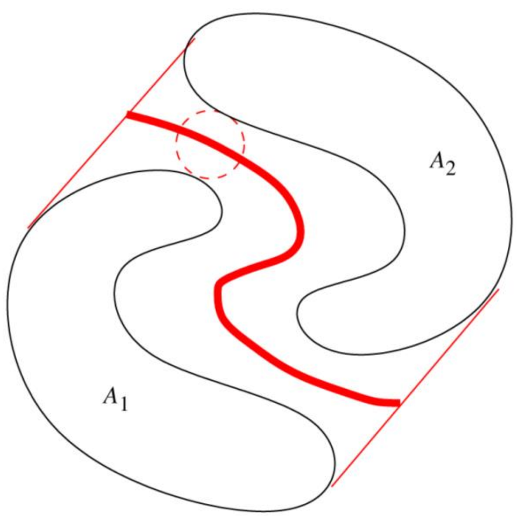


Fig. 14. Definition based on watershed or SKIZ (thick line) in a case where the definition based on simple dilation leads only to the disk limited by the dashed line. (Color version available online at http://ieeexplore.ieee.org.)

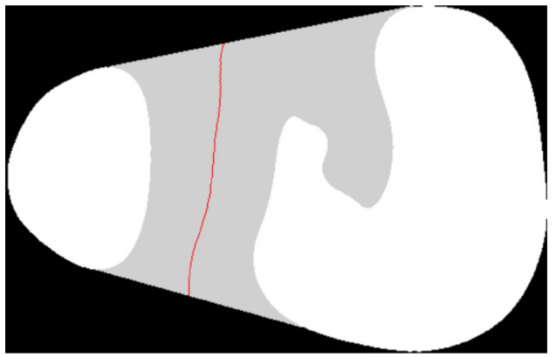


Fig. 15. Definition based on watershed (red line), applied on the white objects and providing the grey area. (Color version available online at http://ieeexplore.ieee.org.)

Morphological Dilations

- Removing non-visible cavities
  - We could use the convex hulls of A_1 and A_2 independently, but this approach cannot handle imbricated objects.

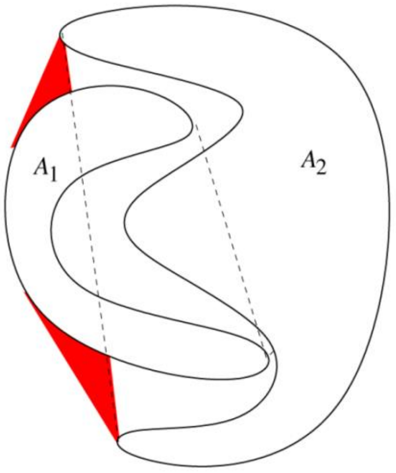


Fig. 16. Working on the convex hulls of the objects does not provide a satisfactory result in the case of imbricated objects: The between area is very reduced (two small parts in red on the left only). (Color version available online at http://ieeexplore.ieee.org.)

Morphological Dilations

- Fuzzy Directional Dilations:
  - Compute the main direction α between A_1 and A_2 as the average or maximum value of the histogram of angles.

$$h_{(A_1,A_2)}(\theta) = \left\{ (a_1,a_2),\ a_1 \in A_1,\ a_2 \in A_2,\ \angle(\overrightarrow{a_1 a_2}, \overrightarrow{u_x}) = \theta \right\}$$

$$H_{(A_1,A_2)}(\theta) = \frac{h_{(A_1,A_2)}(\theta)}{\max_{\theta'} h_{(A_1,A_2)}(\theta')}$$

  - Let D_α denote a dilation in direction α using a fuzzy structuring element, v.

$$D_v(\mu)(x) = \sup_y t[\mu(y), v(x-y)]$$

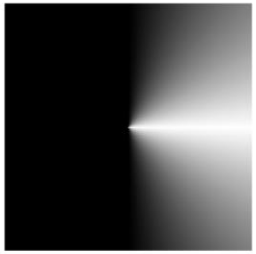


Fig. 17. Example of fuzzy structuring element defined around the horizontal axis.

Morphological Dilations

- Given a fuzzy structuring element in the main direction from A_1 to A_2, we can define the area between A_1 and A_2 as

$$\beta_\alpha(A_1, A_2) = D_\alpha(A_1) \bigcap D_{\pi+\alpha}(A_2) \bigcap A_1^C \bigcap A_2^C$$

- We can consider multiple values for the main direction by defining β *as*

$$\beta(A_1, A_2) = \bigcup_\alpha \beta_\alpha(A_1, A_2) = \bigcup_\alpha (\beta_\alpha \cup \beta_{\alpha+\pi})$$

Morphological Dilations

- Or use the histogram of angles directly as the fuzzy structuring element.

$$v_1(r,\theta) = H_{(A_1,A_2)}(\theta)$$

$$v_2(r,\theta) = H_{(A_1,A_2)}(\theta+\pi) = v_1(r,\theta+\pi)$$

$$\beta_{FDil1}(A_1,A_2) = D_{v_2}(A_1) \cap D_{v_1}(A_2) \cap A_1^C \cap A_2^C$$

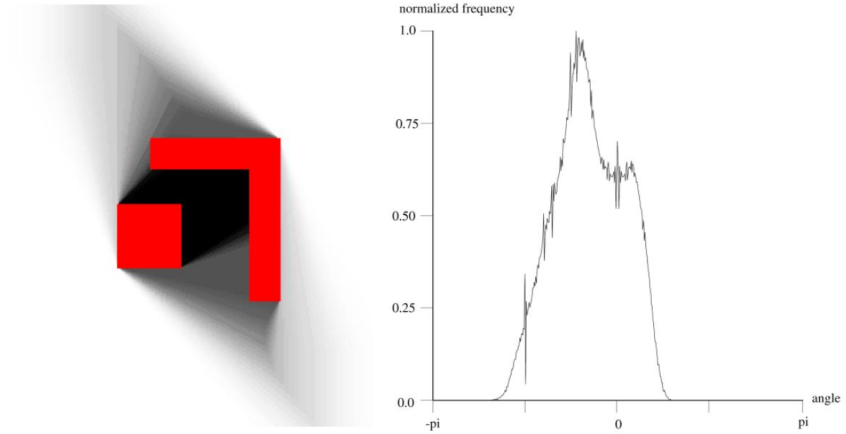


Fig. 18. Definition based on dilation by a structuring element derived from the angle histogram (15). Objects A_1 and A_2 are displayed in red, and the membership values to $\beta(A_1, A_2)$ vary from (white) 0 to (black) 1. The angle histogram is shown on the right. (Color version available online at http://ieeexplore.ieee.org.)

Morphological Dilations

- Alternate definition which removes concavities which are not facing each other

$$\beta_{FDil2}(A_1, A_2) = \left[D_{v_1}(A_1) \cup D_{v_1}(A_2)\right] \cap \left[D_{v_2}(A_1) \cup D_{v_2}(A_2)\right]$$

$$\beta_{FDil3}(A_1, A_2) = D_{v_2}(A_1) \cap D_{v_1}(A_2) \cap A_1^C \cap A_2^C \cap \left[D_{v_1}(A_1) \cap D_{v_1}(A_2)\right]^C \cap \left[D_{v_2}(A_1) \cap D_{v_2}(A_2)\right]^C$$

Fig. 19. Definition based on dilation by a structuring element derived from the angle histogram, with (17). (Color version available online at http://ieeexplore.ieee.org.)

Visibility

- Admissible segments:
  - A segment $[x_1, x_2]$ with x_1 in A_1 and x_2 in A_2 is admissible if it is contained in $A_1^C \cap A_2^C$.
  - $\beta_{Adm}(A_1, A_2)$ is defined as the union of all admissible segments.

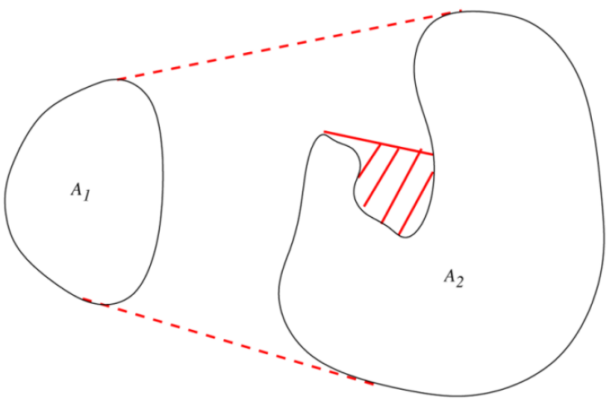


Fig. 20. Example with a nonvisible concavity (dashed area). (Color version available online at http://ieeexplore.ieee.org.)

Visibility

- Fuzzy visibility:
  - Consider a point P and the segments $[a_1, P]$ and $[P, a_2]$ where a_1 and a_2 come from A_1 and A_2 respectively.
  - For each P, find the angle closest to π between any two admissible segments included in $A_1^C \cap A_2^C$

$$\theta_{\min}(P) = \min\{|\pi - \theta|, \theta = \angle([a_1, P],[P, a_2])\}$$

where $[a_1, P]$ and $[P, a_2]$ are semi-admissible segments

$$\beta_{FVisib}(A_1, A_2)(P) = f(\theta_{\min}(P))$$

where $f : [0, \pi] \rightarrow [0,1]$ is a decreasing function

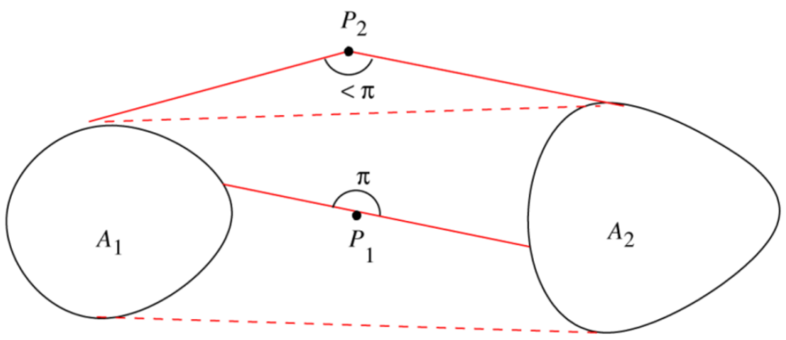


Fig. 21. Illustration of the fuzzy visibility concept. For point P_1 (and any point in the area bounded by the dashed lines), we have $\theta_{\min}(P_1) = 0$ and, therefore, $\beta(P_1) = 1$, while, for point P_2, it is not possible to find two collinear semi-admissible segments from A_1 (respectively, A_2) to P_2; thus, $\theta_{\min}(P_2) > 0$ and $\beta(P_2) < 1$, expressing that P_2 is not completely between A_1 and A_2. (Color version available online at http://ieeexplore.ieee.org.)

Visibility

- Extension to fuzzy objects:
  - Compute the fuzzy inclusion of each segment and intersect with the best angle as before.

$$\mu_{incl}([a_1,a_2]) = \inf_{y\in[a_1,a_2]} \min\left[1-\mu_{A_1}(y), 1-\mu_{A_2}(y)\right]$$

$$\beta_{FVisib}(A_1,A_2)(P) = \max\left\{ \begin{array}{l} t\left[\begin{array}{l} f\left(\left|\theta-\pi\right|\right), a_1 \in Supp(A_1), a_2 \in Supp(A_2), \\ \mu_{incl}([a_1,P]),\ \mu_{incl}([P,a_2]) \end{array} \right] \\ \theta = \angle\left([a_1,P],[P,a_2]\right) \end{array} \right\}$$

Extended Objects

- Objects with different spatial extensions:
  - If A_2 has infinite or near-infinite size with respect to A_1, approximate A_2 by a segment u and dilate A_1 in a direction orthogonal to u, limited to the closest half plane.

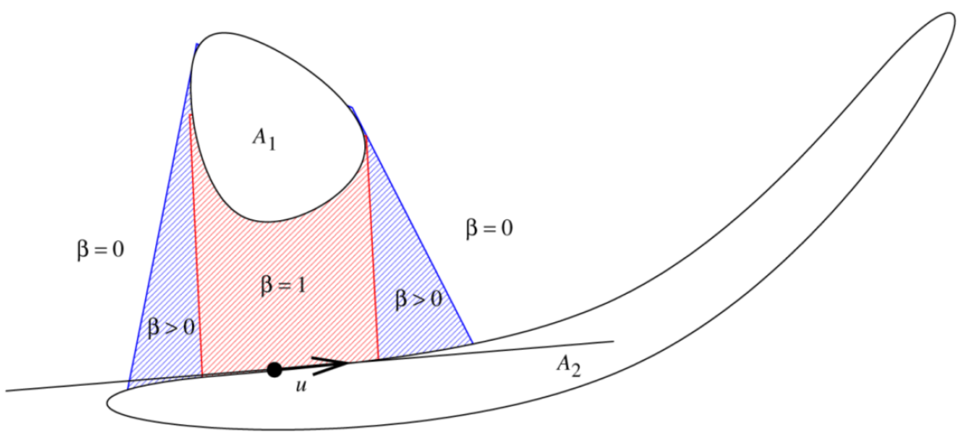


Fig. 22. Illustration of the definition of region β in the case of an extended object (myopic vision). In the areas indicated by $\beta > 0$, the relation is satisfied to some degree between 0 and 1. They can be more or less spread, depending on the structuring element, i.e., on the semantics of the relation. (Color version available online at http://ieeexplore.ieee.org.)

Extended Objects

- Adding a visibility constraint
  - Conjunctively combine projection with admissible segments. Optionally consider only segments orthogonal to u.

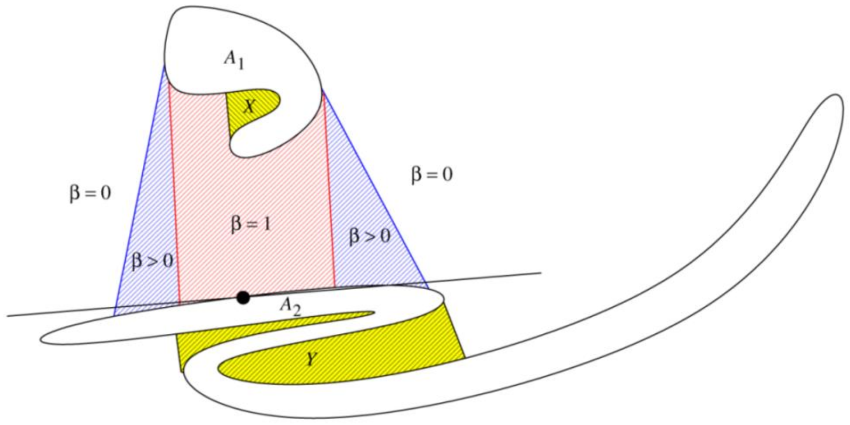


Fig. 23. Example with a concavity in A_1 not visible from A_2 (region X) and an extended object A_2 having concavities not visible from A_1 (Y). (Color version available online at http://ieeexplore.ieee.org.)

Fig. 24. Region β_{Adm} between the two objects on the left, computed by directional fuzzy dilatation, and restricted to the union of admissible segments. The main direction $\vec{u}$ of the extended object is computed locally in a region close to the other object. This is achieved by thresholding a distance map between both objects. The fuzzy structuring element has membership values equal to 1 in the direction orthogonal to $\vec{u}$ and decreasing degrees when going away of this direction. Region β can be more or less spread, depending on the structuring element. (Color version available online at http://ieeexplore.ieee.org.)

Satisfaction Measure

- Once $\beta(A_1,A_2)$ has been determined, we want to know the overlap between β and B.
  - Normalized Intersection:

$$S_1(B,\beta) = \frac{|B \cap \beta|}{|B|}$$

  - Other possible measures:

$$S_2(B,\beta) = 1 - \sup\{\mu_B(x) / \beta(x) = 0\}$$

$$S_3(B,\beta) = \inf_x \min(1 - \mu_B(x) + \beta(x), 1)$$

Satisfaction Measure

- Degree of Inclusion:

$$N(B,\beta) = \inf_x T\left[\beta(x), 1 - \mu_B(x)\right]$$

- Degree of Intersection:

$$\Pi(B,\beta) = \sup_x t\left[\beta(x), \mu_B(x)\right]$$

  - Length of this interval gives information on the ambiguity of the relation.

Satisfaction Measure

- If B is extended, we might want to know if B passes through β.
- Let $R = Supp(\beta) \setminus Core(\beta)$
- R will have two connected components, R_1 and R_2.
- Degree to which B passes through β should be high if B goes at least from a point in R_1 to a point in R_2.

Satisfaction Measure

$$M_1(B,\beta) = \min\left[\sup_{x\in R_1}(B \cap \beta^C)(x), \sup_{x\in R_2}(B \cap \beta^C)(x)\right]$$

$$M_2(B,\beta) = \min\left[\sup_{x\in R_1}(B \cap Core(\beta)^C)(x), \sup_{x\in R_2}(B \cap Core(\beta)^C)(x)\right]$$

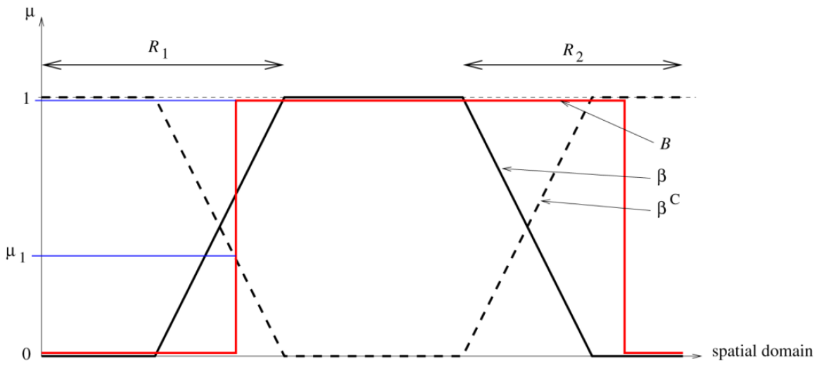


Fig. 25. Example of computation of the degree of satisfaction of the relation in the case of an extended object B (in a 1-D space). Using (29), the obtained value is μ_1, which can be considered a pessimistic evaluation. Using (30), the obtained value is equal to 1, which may better fit the intuitive expectation. (Color version available online at http://ieeexplore.ieee.org.)

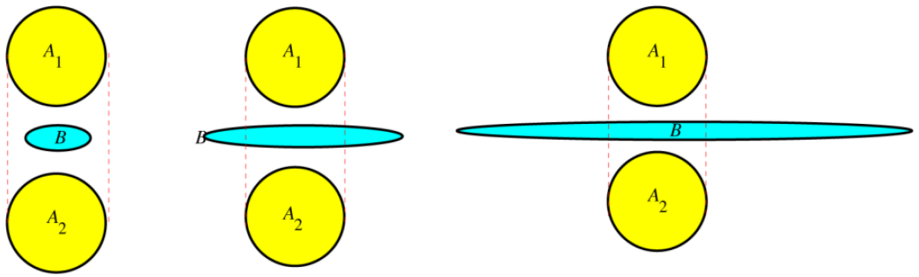


Fig. 26. Three cases where the degree to which B is between A_1 and A_2 may vary, depending on the context and on the definition. (Color version available online at http://ieeexplore.ieee.org.)

Properties

TABLE I
SUMMARY OF THE PROPOSED DEFINITIONS AND THE CASES WHERE THEY APPLY IN A SATISFACTORY WAY

Types of objects → Definitions ↓	Convex	Facing concavities	Complex shape	Fuzzy	Extended
Convex hull β_{CH}	Y	Y	N	Y	N
Dilation β_{Dil}	N	N	N	N	N
Separation by watersheds or SKIZ β_{sep}	Y	Y	N	N	N
Fuzzy directional dilation β_{Fdil1}	Y	Y	N	Y	N
Fuzzy directional dilation β_{FDil2}	Y	Y	N	Y	N
Fuzzy directional dilation β_{FDil3}	Y	Y	Y	Y	N
Admissible segments β_{Adm}	Y	Y	Y	Y	N
Fuzzy visibility β_{Adm}	Y	Y	Y	Y	N
Myopic vision	Y	Y	N	Y	Y
Myopic vision + visibility	Y	Y	Y	Y	Y

Examples

- Anatomical descriptions of brain structures

Fig. 27. Few brain structures (a 2-D slice extracted from a 3-D atlas). (Color version available online at http://ieeexplore.ieee.org.)

TABLE II
FEW RESULTS OBTAINED WITH THE METHOD OF CONVEX HULL (1), FUZZY DIRECTIONAL DILATION [USING (17)] (2), ADMISSIBLE SEGMENTS (3), AND WITH THE FUZZY VISIBILITY APPROACH (4)

A_1	A_2	B	$\frac{\lvert\beta\cap B\rvert}{\lvert B\rvert}$ (1)	$\frac{\lvert\beta\cap B\rvert}{\lvert B\rvert}$ (2)	$\frac{\lvert\beta\cap B\rvert}{\lvert B\rvert}$ (3)	$\frac{\lvert\beta\cap B\rvert}{\lvert B\rvert}$ (4)	$[N, \Pi]$ (1)	$[N, \Pi]$ (4)
CN	LN	IC	0.85	0.84	0.84	0.94	[0, 1]	[0.2, 1]
LV	CG	CC	1.00	0.93	1.00	1.00	[1, 1]	[1, 1]
IFG	SFG	MFG	0.78	0.92	0.76	0.95	[0, 1]	[0.7, 1]
CG	CN	CC	0.88	0.90	0.88	0.97	[0, 1]	[0.6, 1]
CG	CN	LV	0.47	0.63	0.47	0.79	[0, 1]	[0, 1]
IFG	SFG	IC	0.00	0.02	0.00	0.16	[0, 0]	[0, 0.6]
IFG	SFG	LN	0.00	0.00	0.00	0.04	[0, 0]	[0, 0.3]

Examples

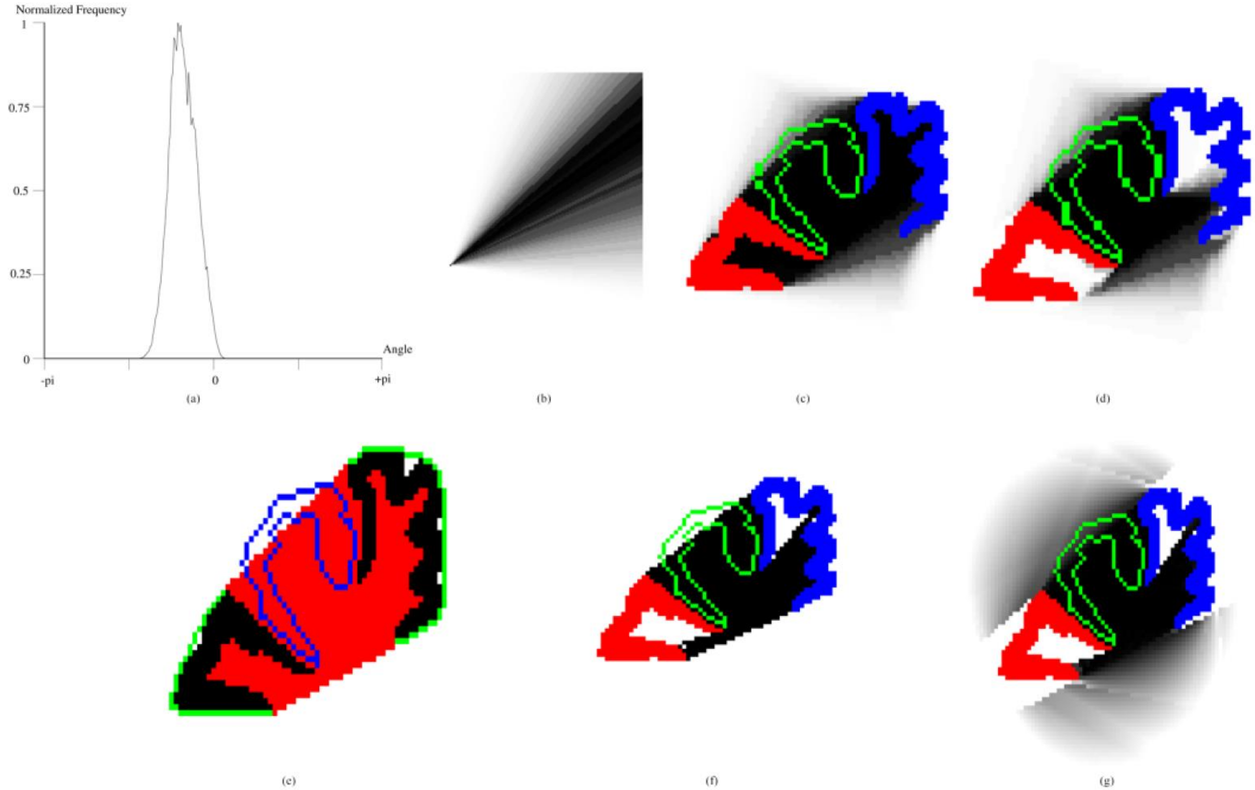


Fig. 28. (a) Angle histogram of objects A_1 and A_2 [superior and inferior frontal gyri, displayed in red and blue in (c))]. (b) Corresponding structuring element ν_1 (ν_2 is its symmetric with respect to the origin). (c) Definition based on fuzzy dilation [with (15)]. Membership values to $\beta(A_1, A_2)$ vary from (white) 0 to (black) 1. The contours of the median frontal gyrus are superimposed in green. (d) Definition based on fuzzy dilation, with (17). (e) Convex hull approach. (f) Definition using the admissible segments. (g) Fuzzy visibility approach. (Color version available online at http://ieeexplore.ieee.org.)

Examples

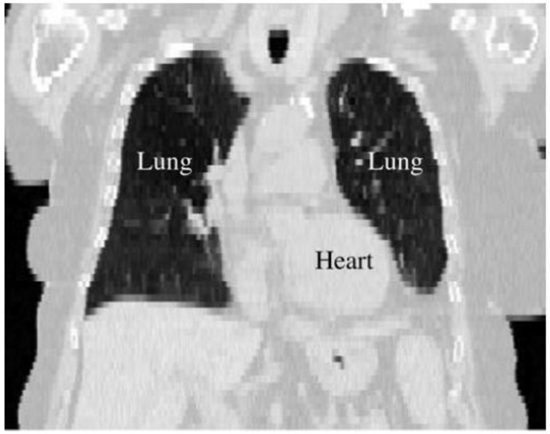


Fig. 29. Coronal slice of a CT image in the thoracic area.

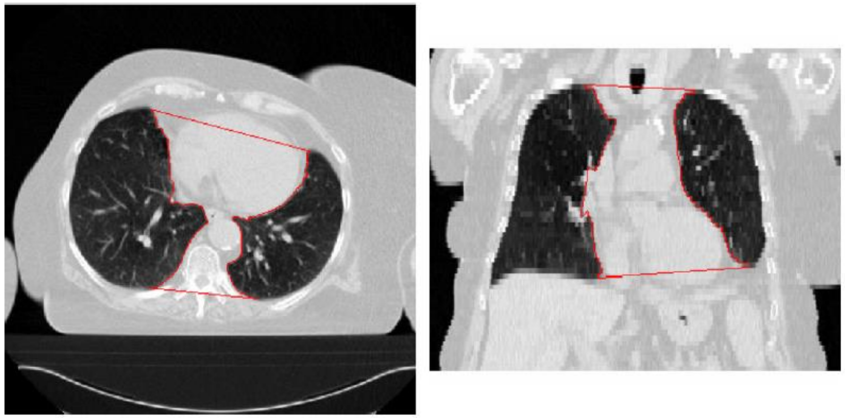


Fig. 30. Contours of β_{Adm} representing the region between the lungs, superimposed on an axial slice and on a coronal slice. (Color version available online at http://ieeexplore.ieee.org.)

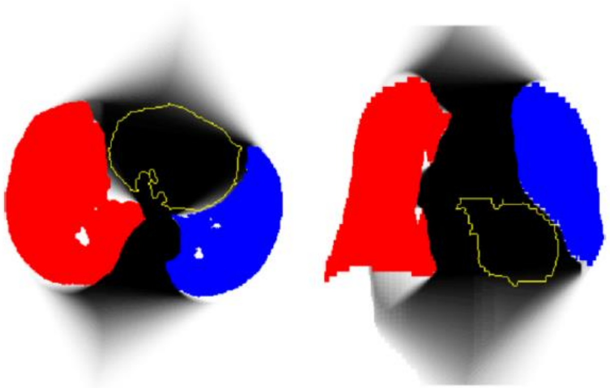


Fig. 31. Fuzzy region β_{FDil3} between the lungs, superimposed on an axial slice and on a coronal slice of the segmented lungs. The contours of the heart and aorta are superimposed too. (Color version available online at http://ieeexplore.ieee.org.)

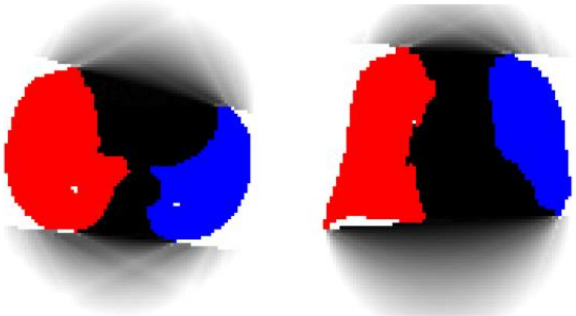


Fig. 32. Fuzzy region β_{FAdm} between the lungs, obtained with the method of semi-admissible segments (fuzzy visibility), superimposed on an axial slice and on a coronal slice of the segmented lungs. (Color version available online at http://ieeexplore.ieee.org.)

Conclusion

- Possible to represent the complex spatial relationship, "between," using mathematical morphology and visibility.
- Many definitions exist, each with individual strengths and weaknesses.
- Pick the representation most appropriate for the application.